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Comment on "Energies of the ground state and first excited 0^+ state in an exactly solvable pairing model" by N. Dinh Dang

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Abstract. We comment on a recent application of the RPA method and its extensions to the case of the two-level pairing model by N. Dinh Dang (Eur. Phys. J. A **16**, 181 (2003)).

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The aim of this Comment is to discuss and explain several aspects related to the derivation and application of the RPA theory in relation with a recent paper [1] by Dinh Dang; there the RPA theory has been applied to the two-level pairing model. We have some criticisms and remarks. We will start with the RPA in the boson formalism as used by N. Dinh Dang [1]. We notice that Dinh Dang uses a quite unconventional and non-systematic boson expansion. Let us restate the standard boson mapping in the case of the two-level pairing model:

$$H = \frac{\epsilon}{2}(\hat{N}_2 - \hat{N}_1) - g\Omega \sum_{jj'} A_j^\dagger A_{j'},\ j, j' = 1, 2\,. \qquad (1)$$

To lowest order one has

$$A_1^\dagger = \frac{1}{\sqrt{\Omega}} \sum_{m>0} a_{1m}^\dagger a_{1\bar{m}}^\dagger \to b_1\,,$$
$$A_2^\dagger = \frac{1}{\sqrt{\Omega}} \sum_{m>0} a_{2m}^\dagger a_{2\bar{m}}^\dagger \to b_2^\dagger\,, \qquad (2)$$

where $b_i^\dagger$, b_i are ideal Bose operators. The occupation number operators obey the exact relations

$$\hat{N}_1 = \sum_m a_{1m}^\dagger a_{1m} \to 2(\Omega - b_1^\dagger b_1)\,,$$
$$\hat{N}_2 = \sum_m a_{2m}^\dagger a_{2m} \to 2b_2^\dagger b_2\,. \qquad (3)$$

Dinh Dang now uses the particle number condition $\langle \hat{N}_1 \rangle + \langle \hat{N}_2 \rangle = \langle \hat{N} \rangle \equiv N \equiv 2\Omega$, which holds if, in the absence of interaction, the lowest level is filled. From the above number condition, one obtains

$$\langle b_1^\dagger b_1 \rangle = \langle b_2^\dagger b_2 \rangle. \qquad (4)$$

Dinh Dang deduces from this relation that $b_1 = b_2 = b$, $b_1^\dagger = b_2^\dagger = b^\dagger$, *i.e.* it is assumed that it is approximately valid to replace the two ideal bosons b_1 and b_2 by the single one b. In the first part of this work we want to study the validity of this single-boson approximation. Keeping the two bosons the pairing Hamiltonian is given to lowest order by

$$H = -\frac{\epsilon}{2} 2(\Omega - b_1^\dagger b_1) + \frac{\epsilon}{2} 2 b_2^\dagger b_2 - g\Omega \sum_{i,j=1,2} b_i^\dagger b_j =$$
$$-\epsilon(\Omega - b_1^\dagger b_1) + \epsilon b_2^\dagger b_2 - g\Omega(b_2^\dagger b_2 + b_1 b_1^\dagger + b_2^\dagger b_1^\dagger + b_1 b_2)\,. \qquad (5)$$

In the single-boson approximation we have

$$H = -\frac{\epsilon}{2} 2(\Omega - b^\dagger b) + \frac{\epsilon}{2} 2 b^\dagger b - g\Omega(b^\dagger b + b b^\dagger + b^\dagger b^\dagger + b b) =$$
$$-\epsilon(\Omega - b^\dagger b) + \epsilon b^\dagger b - g\Omega(b^\dagger b + b b^\dagger + b^\dagger b^\dagger + b b)\,. \qquad (6)$$

Both Hamiltonians can trivially be diagonalized with the help of an RPA (Bogoliubov transformation) among the bosons. We obtain for (5)

$$H = \Omega_1 b_1^\dagger b_1 + \Omega_2 b_2^\dagger b_2 \qquad (7)$$

with

$$\Omega_1 = -g + \sqrt{g+\epsilon}\sqrt{\epsilon + g(1-2\Omega)}\,,$$
$$\Omega_2 = g + \sqrt{g+\epsilon}\sqrt{\epsilon + g(1-2\Omega)}\,, \qquad (8)$$

and for (6)

$$H = \omega b^\dagger b \qquad (9)$$

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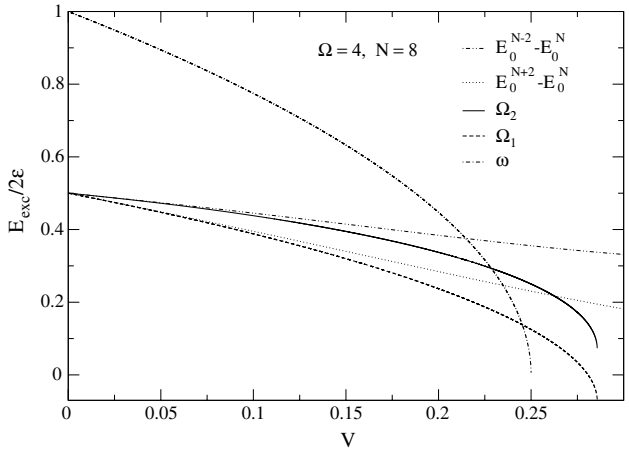

Fig. 1. Excitation energies $E_{\rm exc}$ in the non-superfluid region as function of $V = \Omega/2\epsilon$ described in the text, for particle number $N = 8$. The spin of the levels is $J = 7/2$. The results refer to exact calculations $2\mu^{\pm} = \pm(E_0^{N\pm2} - E_0^N)$ (double-dot–dashed line, dotted line), standard *pp*-RPA eqs. (8) (solid line, dashed line) and *pp*-RPA in the single-boson approximation eq. (10) (dot-dashed line).

with

$$\omega = 2\epsilon\sqrt{1 - \frac{2g\Omega}{\epsilon}}\,. \tag{10}$$

In fig. 1 we have traced these different eigenvalues as a function of $V = g\Omega/2\epsilon$. Also shown are the exact values for the chemical potentials $2\mu^{\pm} = \pm(E_0^{N\pm2} - E_0^N)$ where E_0^N are the exact ground-state energies obtained by diagonalization of the original pairing Hamiltonian. The reason why we compare with $2\mu^{\pm}$ is given by the fact that the eigenvalues of standard *pp*-RPA have to be identified with these chemical potentials (see, *e.g.*, [2–4]). We also see that the eigenvalues Ω_1, Ω_2 follow, at least for small values of V, quite closely the exact values. On the other hand, the single-boson approximation yields a completely erroneous result which seems to have nothing to do with the exact solution.

Let us now comment in the superfluid phase. We note that in the superfluid phase we can calculate in standard QRPA the contribution of the q-term (see eq. (7) in [1]) of the Hamiltonian. Consequently, there is no reason to neglect this term as discussed by N. Dinh Dang [1]. The standard QRPA matrices are well known and given by

$$A_{jj'} = 2\left(E_j + 2q_{jj'}\right)\delta_{jj'} + d_{jj'}\,, \tag{11}$$

$$B_{jj'} = 2\left(1 - \frac{1}{\Omega}\delta_{jj'}\right)h_{jj'}\,, \tag{12}$$

where the quasiparticle energy is $E_j = \sqrt{\left(\epsilon_j - gv_j^2 - \mu\right)^2 + \Delta^2}$ and the gap Δ, as calculated from the BCS equation, includes the self-energy term. We have used the same notation as indicated in the work by Dinh Dang. We remind, shortly, that in this case, the gap is given by

$$\Delta = \sqrt{g^2\Omega^2 - \frac{\xi^2}{4}}, \tag{13}$$

together with

$$u_1^2 = v_2^2 = \frac{1}{2}\left(1 - \frac{\xi}{2g\Omega}\right), \tag{14}$$

$$v_1^2 = u_2^2 = \frac{1}{2}\left(1 + \frac{\xi}{2g\Omega}\right), \tag{15}$$

$$\mu = -\frac{g}{2}, \tag{16}$$

where ξ is defined as $\xi = 2\epsilon\Omega/(2\Omega - 1)$. Using the latter relations we can write

$$u_2^2 - v_2^2 = \frac{\xi}{2g\Omega}; \quad u_1v_1u_2v_2 = u_2^2v_2^2 = \frac{\Delta^2}{4g^2\Omega^2}\,, \tag{17}$$

$$u_2^4 + v_2^4 = \frac{1}{2} + \frac{\xi^2}{8g^2\Omega^2}; \quad v_2^4 = \frac{1}{4} + \frac{\xi^2}{16g^2\Omega^2} - \frac{\xi}{4g\Omega} \tag{18}$$

and we can calculate the different contribution of each term E_j, $q_{jj'}$, $d_{jj'}$, and $h_{jj'}$

$$E_1 = E_2 = g\Omega + \frac{\Delta^2}{4g\Omega^2}; \quad q_{11} = q_{22} = -\frac{\Delta^2}{4g\Omega^2}\,, \tag{19}$$

$$d_{11} = d_{22} = -g\Omega + \frac{\Delta^2}{2g\Omega}; \quad d_{12} = d_{21} = -\frac{\Delta^2}{2g\Omega}\,, \tag{20}$$

$$h_{11} = h_{22} = \frac{\Delta^2}{4g\Omega}; \qquad h_{12} = h_{21} = \frac{g\Omega}{2} - \frac{\Delta^2}{4g\Omega}\,. \tag{21}$$

Therefore, explicitly, the matrix elements are given by

$$A_{11} = A_{22} = g\Omega - \frac{\Delta^2}{2g\Omega^2} + \frac{\Delta^2}{2g\Omega}\,, \tag{22}$$

$$A_{12} = A_{21} = -\frac{\Delta^2}{2g\Omega}\,, \tag{23}$$

$$B_{11} = B_{22} = -\frac{\Delta^2}{2g\Omega^2} + \frac{\Delta^2}{2g\Omega}\,, \tag{24}$$

$$B_{12} = B_{21} = g\Omega - \frac{\Delta^2}{2g\Omega} \tag{25}$$

and the positive eigenvalues of the RPA matrix are given by

$$\Omega_1 = 0, \qquad \Omega_2 = 2\Delta\sqrt{1 - \frac{1}{2\Omega}} \tag{26}$$

in agreement with the result found by, *e.g.*, Hagino and Bertsch [3]. In the case of the fermion formalism of Dinh Dang, to obtain the RPA matrix elements, we have to begin with (see eqs. (57)-(61) in [1])

$$A_{jj'} = 2\left(E_j + 3q_{jj'}\right)\delta_{jj'} + d_{jj'}\,,$$

$$B_{jj'} = 2\left(1 - \frac{1}{\Omega}\delta_{jj'}\right)h_{jj'}\,, \tag{27}$$

where factor 3 is different from the correct factor appearing in (11). Explicitly, in case (27), the QRPA matrices

are given, as follows:

$$A_{11} = A_{22} = g\Omega - \frac{\Delta^2}{g\Omega^2} + \frac{\Delta^2}{2g\Omega}\,, \tag{28}$$

$$A_{12} = A_{21} = -\frac{\Delta^2}{2g\Omega}\,, \tag{29}$$

$$B_{11} = B_{22} = -\frac{\Delta^2}{2g\Omega^2} + \frac{\Delta^2}{2g\Omega}\,, \tag{30}$$

$$B_{12} = B_{21} = g\Omega - \frac{\Delta^2}{2g\Omega}\,, \tag{31}$$

where the gap Δ, as calculated from the BCS equation, includes the self-energy term. The RPA eigenvalues are given by (see eq. (65) in [1])

$$\Omega_1 = \Delta\sqrt{\frac{1}{\Omega}\left(\frac{3\Delta^2}{4g^2\Omega^3} - 1\right)}\,, \tag{32}$$

$$\Omega_2 = 2\Delta\sqrt{\left(1 - \frac{3}{4\Omega}\right)\left(1 - \frac{\Delta^2}{4g^2\Omega^3}\right)}\,. \tag{33}$$

When we neglect the q-term (see eq. (7) in [1]), we obtain

$$A_{11} = A_{22} = g\Omega + \frac{\Delta^2}{2g\Omega^2} + \frac{\Delta^2}{2g\Omega}\,, \tag{34}$$

$$A_{12} = A_{21} = -\frac{\Delta^2}{2g\Omega}\,, \tag{35}$$

$$B_{11} = B_{22} = -\frac{\Delta^2}{2g\Omega^2} + \frac{\Delta^2}{2g\Omega}\,, \tag{36}$$

$$B_{12} = B_{21} = g\Omega - \frac{\Delta^2}{2g\Omega}\,, \tag{37}$$

$$\Omega_1 = \Delta\sqrt{\frac{2}{\Omega}}, \quad \Omega_2 = 2\Delta\sqrt{1 + \frac{\Delta^2}{2g^2\Omega^3}}\,, \tag{38}$$

which leads to RPA eigenvalues different from those given by Dinh Dang for this case. However, if we divide the q-term by a factor of 2, we obtain

$$A_{11} = A_{22} = g\Omega + \frac{\Delta^2}{2g\Omega}\,, \tag{39}$$

$$A_{12} = A_{21} = -\frac{\Delta^2}{2g\Omega}\,, \tag{40}$$

$$B_{11} = B_{22} = -\frac{\Delta^2}{2g\Omega^2} + \frac{\Delta^2}{2g\Omega}\,, \tag{41}$$

$$B_{12} = B_{21} = g\Omega - \frac{\Delta^2}{2g\Omega}\,, \tag{42}$$

which produces the following RPA eigenvalues:

$$\Omega_1 = \Delta\sqrt{\frac{1}{\Omega}\left(1 - \frac{\Delta^2}{4g^2\Omega^3}\right)}\,, \tag{43}$$

$$\Omega_2 = 2\Delta\sqrt{\left(1 - \frac{1}{4\Omega}\right)\left(1 + \frac{\Delta^2}{4g^2\Omega^3}\right)} \tag{44}$$

which coincide, exactly, with the solution given by Dinh Dang in the case where he neglects the q-term. We see that in this case the Goldstone theorem is violated and therefore the particle number symmetry is not restored.

Dinh Dang also treats within the fermion formalism several other approximations which do not produce the Goldstone mode at zero energy. One does not very well understand the aim of these considerations, since any way it is well known [2,3,5] that the Goldstone mode should come at zero energy to restore the particle number symmetry which otherwise is violated.

As a last point we would like to mention that Dinh Dang is superposing addition and removal modes in the non-superfluid phase in eq. (75). This superposition couples, simultaneously, $N \pm 2$ states and therefore, obviously, violates the particle number conservation already in the non-superfluid phase. On the other hand, it is well known [2,3,5] that $pp(hh)$-RPA in the non-superfluid phase perfectly respects the particle number symmetry: the addition mode involves amplitudes $\langle N+2|a^\dagger a^\dagger|N\rangle$ and the removal mode $\langle N-2|aa|N\rangle$. None of these amplitudes violates the particle number.

In short, in this Comment we pointed out a number of shortcomings and inconsistencies in the work by Dinh Dang [1] involving pp-RPA and QRPA in a solvable model.

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